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Maxwell's Equations for Structures with Symmetries

Thomas Weiland and Igor Zagorodnov[1]

Computational Electrodynamics Laboratory, FB 18, Technische Universität Darmstadt, Schloßgartenstrasse 8, Darmstadt D-64289, Germany
E-mail: weiland@temf.de and zagor@temf.de

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Maxwell's equations for structures with arbitrary point symmetry groups are considered. It is shown that an initial boundary value problem for Maxwell's equations in a domain Ω can be reduced to $\sim N$ independent problems in a $1/N$ part of the initial domain, where N is the order of the symmetry group of the domain Ω. This approach allows the developing of effective methods for the numerical solution of problems for structures with symmetries on the basis of any existing numerical algorithm. As virtually all numerical approaches have a more than linear dependence on the computational effort from the dimension of the problem, the approach of solving $\sim N$ problems of size $\sim 1/N$ will result in more-efficient procedures.

Key Words: Maxwell's equations; symmetry; finite groups.

1. INTRODUCTION

The numerical solution of Maxwell's equations in the general form requires prodigious computer resources. However, in the case where the considered structure possesses symmetries it is possible, as is shown in this paper, to considerably reduce the dimension of the problem.

Algorithms for bodies with symmetries based on boundary integral equations for Maxwell's equations in the frequency domain were considered in Refs. [1] and [2]. The algorithm described in this paper allows also the use of finite-difference schemes for the solution of Maxwell's equations in time domain. It is important to note that we do not suppose symmetry of a whole problem. The excitation, boundary, and initial conditions can be arbitrary.

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It is known that for bodies of revolution an initial problem can be reduced to an infinite number of independent problems in a two-dimensional domain Ω_1, being a fundamental domain for the group of rotations C_∞.

From the point of view of group theory this approach is based on the following arguments. If we consider a homeomorphism between group C_∞ and the interval $[0, 2\pi)$, then the group C_∞ possesses the irreducible representations

$$e^{im\varphi}, \quad \varphi \in [0, 2\pi), m = \ldots, -1, 0, 1, \ldots. \tag{1}$$

A function $u(\varphi)$ on the group C_∞ can be represented in the form

$$u(\varphi) = \sum_{m=-\infty}^{m=-\infty} \tilde{u}(m) e^{im\varphi}, \quad \tilde{u}(m) = \int_0^{2\pi} u(\varphi) e^{-im\varphi} \frac{d\varphi}{2\pi}. \tag{2}$$

The relations (2) are Fourier transforms on the group C_∞ [3].

After representation of all functions in form (2) and substitution of them in Maxwell's equation, an infinite number of independent systems of equations is obtained, where each system corresponds to one of the representations (1).

Another type of symmetry described in publications is related to mirror symmetry. However, symmetry of a whole problem is usually considered (excitation and solution also are both symmetric) and the problem is solved in half of the initial domain.

It means that the initial problem for a structure with a group of mirror symmetry I of order 2 is reduced to a problem which corresponds only to one of two representations of group I. It is natural to suggest that in the case of an arbitrary excitation the initial problem can be reduced to two independent problems in half of the whole domain Ω.

The result described above is proven in this paper and generalised for structures with arbitrary symmetry groups.

In Section 2 an algorithm is described for a simple boundary value problem. It clearly shows the basic steps of the algorithm. In Section 3 a boundary value problem for Maxwell's equations for a structure with mirror symmetry is considered in the case of an arbitrary excitation. In Sections 4 and 5 results for an arbitrary symmetry group are presented. Numerical results for a simple structure with the Klein symmetry groups are demonstrated in Section 6.

It is important to note that after the reduction of an initial boundary value problem to a finite number of problems in a fundamental domain, Maxwell's equations retain their form. This allows the use of any existing numerical method for solution of a boundary value problem for Maxwell's equations (a finite-difference time domain method, a finite integration method [4, 5], a finite element method, etc.).

2. AN EXAMPLE OF REDUCTION OF A BOUNDARY VALUE PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION

We consider the following boundary value problem for the ordinary second-order differential equation:

$$\frac{\partial^2}{\partial x^2}u(x) = 6ax + 2b, \quad x \in (-1, 1),$$
$$u(-1) = -a + b - c + d, \quad u(1) = a + b + c + d. \tag{3}$$

This problem has the simple analytical solution

$$u(x) = ax^3 + bx^2 + cx + d,$$

which can be represented in the form

$$u(x) = \begin{cases} \tilde{u}_{\sigma_1}(x) + \tilde{u}_{\sigma_2}(x), & x \in [0, 1], \\ \tilde{u}_{\sigma_1}(-x) - \tilde{u}_{\sigma_2}(-x), & x \in [-1, 0], \end{cases}$$

where $\tilde{u}_{\sigma_1}(x) = bx^2 + d$ and $\tilde{u}_{\sigma_2}(x) = ax^3 + cx$ are solutions of the two boundary value problems

$$\frac{\partial^2}{\partial x^2}\tilde{u}_{\sigma_1}(x) = 2b, \quad x \in (0, 1), \quad \frac{\partial}{\partial x}\tilde{u}_{\sigma_1}(0) = 0, \quad \tilde{u}_{\sigma_1}(1) = b + d, \tag{4}$$

$$\frac{\partial^2}{\partial x^2}\tilde{u}_{\sigma_2}(x) = 6ax, \quad x \in (0, 1), \quad \tilde{u}_{\sigma_2}(0) = 0, \quad \tilde{u}_{\sigma_2}(1) = a + c. \tag{5}$$

Hence, the initial problem (3) can be reduced to two problems, (4) and (5), defined in half of the original domain. Note that the form of the differential operator does not change, because of its invariance to the mirror transformation.

In order to describe a general algorithm for problems with symmetries, we show how problems (4) and (5) can be derived in a systematic approach.

The domain $\Omega = \{x : x \in [-1, 1]\}$ possesses the mirror symmetry group $I = \{\tau_1, \tau_2\}$, where $\{\tau_1 = 1, \tau_2 = -1\}$ is a realisation of the mirror group in the given coordinate system. Consequently, the representation

$$\Omega = \Omega_1 \bigcup \tau_2 \Omega_1,$$

where $\Omega_1 = [0, 1]$ is a fundamental domain of the group I acting on Ω, holds true.

The solution of Eq. (3) belongs to the space $C^1(\Omega)$ of differentiable functions. Hence, the relations

$$u(-0) = u(+0), \tag{6}$$

$$\frac{\partial}{\partial x}u(-0) = \frac{\partial}{\partial x}u(+0) \tag{7}$$

are fulfilled.

We introduce a mapping θ of the space $C^1(\Omega)$ into a space $L(I, C^1(\Omega_1))$ of functions on the set $I \times \Omega_1$ with the conditions

$$\hat{u}(\tau_1, 0) = \hat{u}(\tau_2, 0), \quad \frac{\partial}{\partial x}\hat{u}(\tau_1, 0) = -\frac{\partial}{\partial x}\hat{u}(\tau_2, 0).$$

The mapping θ is defined by the relation

$$\hat{u}(\tau_i, x) = [\theta u](\tau_i, x) = [\tau_i^* u](x) = u(\tau_i x).$$

Hence, the mapping θ is biunique and we are led to the problem of determining the function $\hat{u}(\tau_i, x)$.

The function $\hat{u}(\tau_i, x)$ can be represented as

$$\hat{u}(\tau_i, x) = \tilde{u}_{\sigma_1}(x)\sigma_1(\tau_i) + \tilde{u}_{\sigma_2}(x)\sigma_2(\tau_i),$$

where $\{\sigma_i\}$ are irreducible representations of the group I,

$$\sigma_1(\tau_1) = \sigma_1(\tau_2) = 1, \quad \sigma_2(\tau_1) = 1, \sigma_2(\tau_2) = -1,$$

and $\tilde{u}_{\sigma_i}(x)$ can be found through Fourier transform on the group I,

$$\tilde{u}_{\sigma_i}(x) = \frac{1}{2}\left(\hat{u}(\tau_1, x)\sigma_i\left(\tau_1^{-1}\right) + \hat{u}(\tau_2, x)\sigma_i\left(\tau_2^{-1}\right)\right). \tag{8}$$

By the mapping θ the right hand side, $f(x)$, of Eq. (3) is transformed into a function

$$\hat{f}(\tau_i, x) = \begin{cases} 6ax + 2b, & \tau_i = \tau_1, \\ -6ax + 2b, & \tau_i = \tau_2, \end{cases} \quad x \in \Omega_1.$$

Hence, applying relation (8) we obtain

$$\tilde{f}_{\sigma_1}(x) = 2b, \quad \tilde{f}_{\sigma_2}(x) = 6ax, \quad x \in \Omega_1. \tag{9}$$

Analogously, for the boundary function $g(x)$, $x \in \{-1, 1\}$, we have

$$\begin{gathered} \hat{g}(\tau_i, 1) = \begin{cases} a + b + c + d, & \tau_i = \tau_1, \\ -a + b - c + d, & \tau_i = \tau_1, \end{cases} \\ \tilde{g}_{\sigma_1}(1) = b + d, \quad \tilde{g}_{\sigma_2}(1) = a + c. \end{gathered} \tag{10}$$

In order to impose boundary conditions for the functions $\tilde{u}_{\sigma_i}(x)$ at $x = 0$ we make use of the relations of continuity (6) and (7) and obtain

$$\hat{u}(\tau_1, 0) = \hat{u}(\tau_2, 0), \quad \tilde{u}_{\sigma_1}(0) + \tilde{u}_{\sigma_2}(0) = \tilde{u}_{\sigma_1}(0) - \tilde{u}_{\sigma_2}(0), \quad \tilde{u}_{\sigma_2}(0) = 0; \tag{11}$$

$$\begin{gathered} \frac{\partial}{\partial x}\hat{u}(\tau_1, 0) = -\frac{\partial}{\partial x}\hat{u}(\tau_2, 0), \quad \frac{\partial}{\partial x}\left(\tilde{u}_{\sigma_1}(0) + \tilde{u}_{\sigma_2}(0)\right) = -\frac{\partial}{\partial x}\left(\tilde{u}_{\sigma_1}(0) - \tilde{u}_{\sigma_2}(0)\right), \\ \frac{\partial}{\partial x}\tilde{u}_{\sigma_1}(0) = 0. \end{gathered} \tag{12}$$

Taking into account the invariance of the operator of Eq. (3) with respect to the group I, relations (9)–(12) yield two independent boundary value problems, (4) and (5), in half domain.

3. REDUCTION OF A BOUNDARY VALUE PROBLEM FOR MAXWELL'S EQUATIONS IN A DOMAIN WITH MIRROR SYMMETRY

In the following the simplest case of a structure with mirror symmetry is considered. Below, the time dependence of functions is not noted because transformations from a group G have no impact on the time coordinate.

The operator of the system of Maxwell's equations

$$
\begin{gathered}
\nabla \times H - \frac{\partial D}{\partial t} = j, \quad \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad x \in \Omega, \\
\nabla \cdot D = \rho, \quad \nabla \cdot B = 0, \\
D = \varepsilon E, \quad B = \mu H
\end{gathered}
\tag{13}
$$

is invariant with respect to a symmetry group G of the domain Ω, if material distributions ε and μ possess symmetries of the group G,

$$
\varepsilon(x) = \varepsilon(\tau_k x), \quad \mu(x) = \mu(\tau_k x), \quad \forall \tau_k \in G. \tag{14}
$$

To define a boundary value problem it is necessary, for example, to impose boundary values for the tangential component of the electric field

$$
E_t = F(x), \quad x \in \partial\Omega, \tag{15}
$$

where $\partial\Omega$ is a boundary of the domain Ω. The initial conditions are assumed homogeneous.

The domain Ω with mirror symmetry can be decomposed as

$$
\Omega = \Omega_1 + \tau_2 \Omega_1,
$$

where τ_2 is a mirror transformation from the mirror symmetry group $I = \{\tau_1, \tau_2\}$.

We show that for arbitrary functions j, ρ, F this problem can be reduced to two boundary value problems in the half domain Ω_1 of the initial domain Ω.

A solution of the problem (13)–(15) can be presented in the form

$$
\begin{aligned}
E(x) &= \begin{cases} \tilde{E}_{\sigma_1}(x) + \tilde{E}_{\sigma_2}(x), & x \in \Omega_1, \\ \tau_2\left(\tilde{E}_{\sigma_1}\left(\tau_2^{-1}x\right) - \tilde{E}_{\sigma_2}\left(\tau_2^{-1}x\right)\right), & x \in \tau_2\Omega_1, \end{cases} \\
H(x) &= \begin{cases} \tilde{H}_{\sigma_1}(x) + \tilde{H}_{\sigma_2}(x), & x \in \Omega_1, \\ -\tau_2\left(\tilde{H}_{\sigma_1}\left(\tau_2^{-1}x\right) - \tilde{H}_{\sigma_2}\left(\tau_2^{-1}x\right)\right), & x \in \tau_2\Omega_1, \end{cases}
\end{aligned}
\tag{16}
$$

where τ_2 is a mirror transformation in the space R^3. For example, if the mirror plane coincides with the plane OXY, then

$$
\tau_2 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}.
$$

To obtain the fields E, H it is necessary only to add and subtract components of the fields $\tilde{E}_{\sigma_1}$, $\tilde{H}_{\sigma_1}$ and $\tilde{E}_{\sigma_2}$, $\tilde{H}_{\sigma_2}$.

The boundary value problem for fields $\tilde{E}_{\sigma_1}$, $\tilde{H}_{\sigma_1}$ reads

$$
\begin{gathered}
\nabla \times \tilde{H}_{\sigma_1} - \frac{\partial \tilde{D}_{\sigma_1}}{\partial t} = \tilde{j}_{\sigma_1}, \quad \nabla \times \tilde{E}_{\sigma_1} + \frac{\partial \tilde{B}_{\sigma_1}}{\partial t} = 0, \quad x \in \Omega_1, \\
\nabla \cdot \tilde{D}_{\sigma_1} = \tilde{\rho}_{\sigma_1}, \quad \nabla \cdot \tilde{B}_{\sigma_1} = 0, \\
\tilde{D}_{\sigma_1} = \varepsilon \tilde{E}_{\sigma_1}, \quad \tilde{B}_{\sigma_1} = \mu \tilde{H}_{\sigma_1}, \\
\left(\tilde{E}_{\sigma_1}\right)_t = \tilde{F}_{\sigma_1}(x), \quad x \in \partial\Omega_1 \backslash S, \\
\left(\tilde{H}_{\sigma_1}\right)_t = 0, \quad x \in S,
\end{gathered}
\tag{17}
$$

where S is the part of the boundary $\partial\Omega_1$, which coincides with the mirror plane, and

$$\tilde{j}_{\sigma_1}(x) = \frac{1}{2}\left(j(x) + \tau_2^{-1} j(\tau_2 x)\right), \quad x \in \Omega_1,$$
$$\tilde{F}_{\sigma_1}(x) = \frac{1}{2}\left(F(x) + \tau_2^{-1} F(\tau_2 x)\right), \quad x \in \partial\Omega_1 \backslash S,$$
$$\tilde{\rho}_{\sigma_1}(x) = \frac{1}{2}(\rho(x) - \rho(\tau_2 x)), \quad x \in \Omega_1.$$

Similarly, for the fields $\tilde{E}_{\sigma_2}$, $\tilde{H}_{\sigma_2}$ we have

$$\begin{gathered}
\nabla \times \tilde{H}_{\sigma_2} - \frac{\partial \tilde{D}_{\sigma_2}}{\partial t} = \tilde{j}_{\sigma_2}, \quad \nabla \times \tilde{E}_{\sigma_2} + \frac{\partial \tilde{B}_{\sigma_2}}{\partial t} = 0, \quad x \in \Omega_1, \\
\nabla \cdot \tilde{D}_{\sigma_2} = \tilde{\rho}_{\sigma_2}, \quad \nabla \cdot \tilde{B}_{\sigma_2} = 0, \\
\tilde{D}_{\sigma_2} = \varepsilon \tilde{E}_{\sigma_2}, \quad \tilde{B}_{\sigma_2} = \mu \tilde{H}_{\sigma_2}, \\
\left(\tilde{E}_{\sigma_2}\right)_t = \tilde{F}_{\sigma_2}(x), \quad x \in \partial\Omega_1 \backslash S, \\
\left(\tilde{E}_{\sigma_2}\right)_t = 0, \quad x \in S,
\end{gathered} \tag{18}$$

where

$$\tilde{j}_{\sigma_2}(x) = \frac{1}{2}\left(j(x) - \tau_2^{-1} j(\tau_2 x)\right), \quad x \in \Omega_1,$$
$$\tilde{F}_{\sigma_2}(x) = \frac{1}{2}\left(F(x) - \tau_2^{-1} F(\tau_2 x)\right), \quad x \in \partial\Omega_1 \backslash S,$$
$$\tilde{\rho}_{\sigma_2}(x) = \frac{1}{2}(\rho(x) + \rho(\tau_2 x)), \quad x \in \Omega_1.$$

The problems (17) and (18) are defined in half of the initial domain and their operators differ by the boundary condition on the boundary S, which coincides with the symmetry plane. The initial conditions for both problems remain homogeneous.

In the following, obtaining the boundary value problems (17) and (18) is described.

The solution of an initial boundary value problem has continuous tangential components in points of the set S of the domain Ω. Hence, it satisfies the relations

$$\lim_{\substack{x\to S\\ x\in D_1}} E_t(x) = \lim_{\substack{x\to S\\ x\in \tau_2 D_1}} E_t(x), \tag{19}$$

$$\lim_{\substack{x\to S\\ x\in D_1}} H_t(x) = \lim_{\substack{x\to S\\ x\in \tau_2 D_1}} H_t(x). \tag{20}$$

We introduce a mapping θ of the space of the vector fields in the domain Ω into a space of the vector fields on the set $I \times \Omega_1$, as defined by the relation

$$\hat{P}(\tau_i, x) = [\theta P](\tau_i, x) = \alpha\left[\left(\tau_i^{-1}\right)_* P\right](x) = \alpha \tau_i^{-1} P(\tau_i x),$$

where $\alpha = 1$ if P is a vector, and α is equal to the determinant of the matrix τ_i if P is a pseudovector [6]. Here, $\det(\tau_1) = 1$, $\det(\tau_2) = -1$. In the boundary value problem (13)–(15) E, D, j, F are vectors, other functions are pseudovectors, and ε, μ are scalars, for which the mapping θ is defined in Section 1. The charge density ρ is a pseudoscalar with mapping θ defined by the relation

$$\hat{\rho}(\tau_i, x) = [\theta\rho](\tau_i, x) = \det(\tau_i)[\tau_i^* \rho](x) = \det(\tau_i)\rho(\tau_i x).$$

Hence, we are led to the problem of determining the fields $\hat{E}(\tau_i, x)$ and $\hat{H}(\tau_i, x)$.

We represent all functions of Eqs. (13) in the form

$$\hat{P}(\tau_i, x) = \tilde{P}_{\sigma_1}(x)\sigma_1(\tau_i) + \tilde{P}_{\sigma_2}(x)\sigma_2(\tau_i), \quad x \in \Omega_1,$$

where $\{\sigma_i\}$ are irreducible representations of the group I, and functions $\tilde{P}_{\sigma_i}(x)$ are calculated through Fourrier transform on the group I:

$$\tilde{P}_{\sigma_i}(x) = \frac{1}{2}\left(\hat{P}(\tau_1, x)\sigma_i\left(\tau_1^{-1}\right) + \hat{P}(\tau_2, x)\sigma_i\left(\tau_2^{-1}\right)\right), \quad x \in \Omega_1.$$

Similarly, for the boundary function $F(x)$, $x \in \partial\Omega$, we obtain

$$\tilde{F}_{\sigma_i}(x) = \frac{1}{2}\left(\hat{F}(\tau_1, x)\sigma_i\left(\tau_1^{-1}\right) + \hat{F}(\tau_2, x)\sigma_i\left(\tau_2^{-1}\right)\right), \quad x \in \partial\Omega_1 \backslash S.$$

For imposing boundary conditions on the boundary S we use the relations (19) and (20). We obtain

$$\begin{gathered}
\hat{E}_t(\tau_1, x) = \tau_2^{-1}\hat{E}_t(\tau_2, x), \quad \left(\tilde{E}_{\sigma_1}\right)_t(x) + \left(\tilde{E}_{\sigma_2}\right)_t(x) = \tau_2^{-1}\left(\left(\tilde{E}_{\sigma_1}\right)_t(x) - \left(\tilde{E}_{\sigma_2}\right)_t(x)\right), \\
\left(\tilde{E}_{\sigma_2}\right)_t(x) = 0; \\
\hat{H}_t(\tau_1, x) = -\tau_2^{-1}\hat{H}_t(\tau_2, x), \quad \left(\tilde{H}_{\sigma_1}\right)_t(x) + \left(\tilde{H}_{\sigma_2}\right)_t(x) = -\tau_2^{-1}\left(\left(\tilde{H}_{\sigma_1}\right)_t(x) - \left(\tilde{H}_{\sigma_2}\right)_t(x)\right), \\
\left(\tilde{H}_{\sigma_1}\right)_t(x) = 0.
\end{gathered}$$

Taking into account the invariance of the operator of Eqs. (13) with respect to the group I, the above relations yield two boundary value problems, (17) and (18), in half of the whole domain.

In conclusion, we note that the reduction described above is possible only if the operator of a boundary value problem is invariant with respect to a symmetry group of the domain. It means that we require not only invariance of the operator of a system of differential equations, but also invariance of the operator of boundary conditions, i.e., in points of boundary which lie on the same orbit the type of boundary operator has to be the same. However, the right hand side of the system of equations, the initial values, and the boundary values are arbitrary and are not restricted to symmetric ones.

4. REDUCTION OF A BOUNDARY VALUE PROBLEM FOR MAXWELL'S EQUATIONS FOR A DOMAIN WITH A COMMUTATIVE SYMMETRY GROUP

In the following we extend the above discussion on a system with mirror symmetry to consider the common case of structures with arbitrary commutative groups.

Suppose that a domain Ω, in which a boundary value problem for Maxwell's equations is considered, possesses a commutative symmetry group G of order N. We show that the problem can be reduced to N independent problems for Maxwell's equations in a fundamental domain Ω_1, which is only the $1/N$ part of the whole domain Ω.

For the domain Ω with the symmetry group G the following representation holds true:

$$\Omega = \bigcup_{\tau_i \in G} \tau_i \Omega_1.$$

We introduce a mapping as

$$\hat{P}(\tau_i, x) = [\theta P](\tau_i, x) = \alpha[(\tau_i)^* P](x),$$

where $(\tau_i)^*$ is the operation of transferring scalar and vector fields from the domain $\tau_i \Omega_1$ to the domain Ω_1 [7],

$$\begin{aligned} [(\tau_i)^* f](x) &= f(\tau_i x), \\ [(\tau_i)^* V](x) &= \left[\left(\tau_i^{-1}\right)_* V\right](x) = \tau_i^{-1} V(\tau_i x), \end{aligned}$$

and $\alpha = 1$ if f is a scalar function (V is a vector function), and α is equal to the determinant of the matrix τ_i if f is a pseudoscalar function (V is a pseudovector function) [6]. Here, $\det(\tau_i) = 1$ if τ_i is a pure rotation, and $\det(\tau_i) = -1$ if τ_i is an inversion rotation.

For the fields E, H we have

$$\begin{aligned} \hat{E}(\tau_i, x) &= \tau_i^{-1} E(\tau_i x), \quad x \in \Omega_1, \\ \hat{H}(\tau_i, x) &= \det(\tau_i) \tau_i^{-1} H(\tau_i x), \end{aligned}$$

and inverse relations

$$\begin{aligned} E(x) &= \tau_i \hat{E}\left(\tau_i, \tau_i^{-1} x\right), \quad x \in \tau_i \Omega_1, \\ H(x) &= \det(\tau_i) \tau_i \hat{H}\left(\tau_i, \tau_i^{-1} x\right). \end{aligned}$$

Considering $\hat{P}(\tau_i, x)$ as a function on the group G we can write [3]

$$\begin{aligned} \hat{P}(\tau_i, x) &= \frac{1}{N} \sum_{j=1}^{N} \tilde{P}_{\sigma_j}(x) \sigma_j(\tau_i), \\ \tilde{P}_{\sigma_j}(x) &= \sum_{i=1}^{N} \hat{P}(\tau_i, x) \sigma_j\left(\tau_i^{-1}\right). \end{aligned} \tag{21}$$

Through representation of all functions by Fourier series (21) on the group G, we obtain N independent boundary value problems for Maxwell's equations with the same type of boundary conditions on the part of the boundary $T = \partial\Omega_1 \cap \partial\Omega$ of the domain Ω_1.

However, for a complete definition of boundary value problems it is necessary to impose boundary conditions on the boundary part $S = \partial\Omega_1 \backslash T$, which we denote the "cutting boundary." It is shown below that in the case of a commutative group it is possible to obtain disconnected boundary conditions.

The boundary conditions on the cutting boundary $S = \partial\Omega_1 \backslash T$ of the domain Ω_1 are obtained from the condition of continuity of the tangential components of the field E and H in the transition through the surface S.

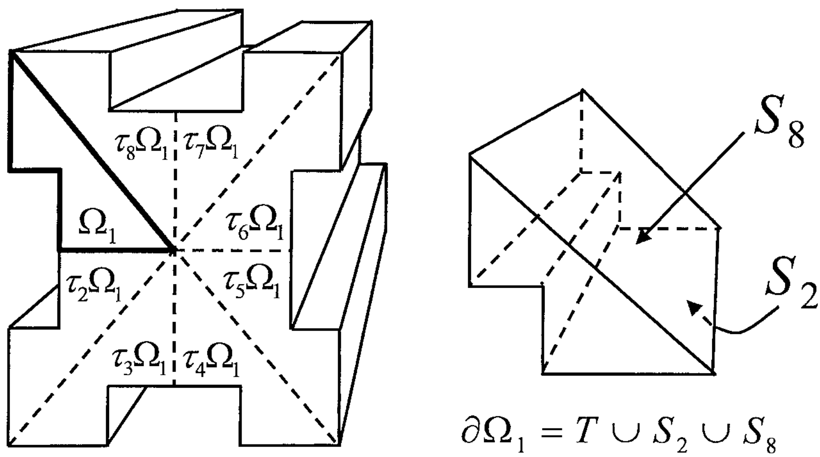

FIG. 1. An example of a structure with a noncommutative symmetry group.

From the sets $S_k = \tau_k\Omega_1 \cap \Omega_1$ we select the ones for which the dimension (as manifold in three-dimensional space) is equal to 2. From their indexes we compose a new set *Ind* (for example, see Fig. 1). It is easy to show that the set *Ind* contains no more than three elements.

The boundary S can be represented in the form

$$S = \bigcup_{k \in Ind} S_k.$$

For the fields E, H in the domain Ω the relations

$$\begin{aligned}
\lim_{\substack{x \to \tau_i S_k \\ x \in \tau_i D_1}} E_t &= \lim_{\substack{x \to \tau_i S_k \\ x \in \tau_i \tau_k D_1}} E_t, \\
\lim_{\substack{x \to \tau_i S_k \\ x \in \tau_i D_1}} H_t &= \lim_{\substack{x \to \tau_i S_k \\ x \in \tau_i \tau_k D_1}} H_t, \quad k \in Ind, \forall \tau_i \in G,
\end{aligned}$$

hold true and, consequently,

$$\begin{aligned}
(\hat{E}(\tau_i, x))_t &= \left(\tau_k \hat{E}\left(\tau_i\tau_k, \tau_k^{-1}x\right)\right)_t, \quad \forall \tau_i \in G, \\
(\hat{H}(\tau_i, x))_t &= \det(\tau_k)\left(\tau_k \hat{H}\left(\tau_i\tau_k, \tau_k^{-1}x\right)\right)_t, \quad x \in S_k, k \in Ind.
\end{aligned} \tag{22}$$

Representing the fields $\hat{E}$, $\hat{H}$ in form (21) and using the orthogonality of characters we obtain the relations

$$\begin{aligned}
\left(\tilde{E}_{\sigma_j}(x)\right)_t &= \left(\tau_k \tilde{E}_{\sigma_j}\left(\tau_k^{-1}x\right)\right)_t \sigma_j(\tau_k), \\
\left(\tilde{H}_{\sigma_j}(x)\right)_t &= \det(\tau_k)\left(\tau_k \tilde{H}_{\sigma_j}\left(\tau_k^{-1}x\right)\right)_t \sigma_j(\tau_k), \quad x \in S_k, k \in Ind.
\end{aligned} \tag{23}$$

The conditions (23) are disconnected concerning indexes $j = 1, 2, \ldots, N$, so that we obtain N independent boundary value problems for Maxwell's equations. Note that the operator of the system of Maxwell's equations does not change its form because it is invariant to transformations of the group G.

As an example, the problem (13)–(15) with homgeneous initial conditions in a domain Ω with a commutative symmetry group of an order N can be reduced in $1/N$ part of the domain Ω to the N independent problems

$$
\begin{gathered}
\nabla \times \tilde{H}_{\sigma_j} - \frac{\partial \tilde{D}_{\sigma_j}}{\partial t} = \tilde{j}_{\sigma_j}, \quad \nabla \times \tilde{E}_{\sigma_j} + \frac{\partial \tilde{B}_{\sigma_j}}{\partial t} = 0, \quad x \in \Omega_1, \\
\nabla \cdot \tilde{D}_{\sigma_j} = \tilde{\rho}_{\sigma_j}, \quad \nabla \cdot \tilde{B}_{\sigma_j} = 0, \\
\tilde{D}_{\sigma_j} = \varepsilon \tilde{E}_{\sigma_j}, \quad \tilde{B}_{\sigma_j} = \mu \tilde{H}_{\sigma_j}, \\
\left(\tilde{E}_{\sigma_j}\right)_t = \tilde{F}_{\sigma_j}(x), \quad x \in \partial\Omega_1 \backslash S, \\
\left(\tilde{E}_{\sigma_j}(x)\right)_t = \left(\tau_k \tilde{E}_{\sigma_j}\left(\tau_k^{-1} x\right)\right)_t \sigma_j(\tau_k), \\
\left(\tilde{H}_{\sigma_j}(x)\right)_t = \det(\tau_k)\left(\tau_k \tilde{H}_{\sigma_j}\left(\tau_k^{-1} x\right)\right)_t \sigma_j(\tau_k), \quad x \in S_k, \, k \in Ind,
\end{gathered}
\tag{24}
$$

where $j = 1, 2, \ldots, N$, $S = \bigcup_{k \in Ind} S_k$, and the initial conditions remain homogeneous.

Generalising the described algorithm to the case of inhomogeneous initial conditions and to another type of boundary conditions in now a straightforward task.

5. REDUCTION OF A BOUNDARY VALUE PROBLEM FOR MAXWELL'S EQUATIONS FOR A DOMAIN WITH AN ARBITRARY SYMMETRY GROUP

Structures with a commutative group were considered above. In this part the case of a noncommutative symmetry group is treated.

Suppose that a domain Ω, in which a boundary value problem for Maxwell's equations is considered, possesses a noncommutative symmetry group G of order N. It is known that the group G possesses $p \leq N$ irreducible representations $\{\sigma_j\}$, and that the dimension $d(\sigma_j)$ of some representations in the case of a noncommutative group is more than 1.

We show that the boundary value problem for Maxwell's equations can be reduced to $s = \sum_{j=1}^{p} d(\sigma_j)$, $s \leq N$, independent problems in a fundamental domain Ω_1, which is only $1/N$ part of the whole domain Ω.

The formulas (21) we write in a more common form,

$$
\begin{aligned}
\hat{P}(\tau_i, x) &= \frac{1}{N} \sum_{j=1}^{p} d(\sigma_j) tr\left(\tilde{P}_{\sigma_j}(x) \sigma_j(\tau_i)\right), \\
\tilde{P}_{\sigma_j}(x) &= \sum_{i=1}^{N} \hat{P}(\tau_i, x) \sigma_j\left(\tau_i^{-1}\right),
\end{aligned}
\tag{25}
$$

where p is a number of irreducible representations of the group G. Note that in the case of $d(\sigma_j) > 1$ the elements of the representation $\sigma_j(\tau_i)$ are matrixes on the order of $d(\sigma_j)$ and, consequently, the values $\tilde{P}_{\sigma_j}$ will be also matrixes on the order of $d(\sigma_j)$.

Proceeding as in the case of commutative group and representing all function in the form of Fourier series (25) on the group G, we obtain N independent boundary value problems for Maxwell's equations with the same type of boundary conditions on the boundary part $T = \partial\Omega_1 \cap \partial\Omega$ of the domain Ω_1.

For a complete definition of boundary value problems it is, however, necessary to impose boundary conditions on the boundary part $S=\partial\Omega_1\backslash T$, and as shown below, some problems will be coupled in the noncommutative case.

First, as in the case of a commutative group, we are led to relations (22). Representing $\hat{E}$ in form (25), we obtain

$$\sum_{j=1}^{p} d(\sigma_j)tr\Big(\big(\tilde{E}_{\sigma_j}(x)\big)_t\sigma_j(\tau_i)\Big) = \sum_{j=1}^{p} d(\sigma_j)tr\Big(\big(\tau_k\tilde{E}_{\sigma_j}\big(\tau_k^{-1}x\big)\big)_t\sigma_j(\tau_i\tau_k)\Big),$$
$$\forall\tau_i\in G,\ x\in S_k,\ k\in Ind.$$

Using the identity $tr(AB)=tr(BA)$, which holds for arbitrary matrixes A and B, we obtain conditions

$$\big(\tilde{E}_{\sigma_j}(x)\big)_t=\sigma_j(\tau_k)\big(\tau_k\tilde{E}_{\sigma_j}\big(\tau_k^{-1}x\big)\big)_t,\quad x\in S_k,\ k\in Ind. \tag{26}$$

Analogously, for the field H we obtain

$$\big(\tilde{H}_{\sigma_j}(x)\big)_t=\det(\tau_k)\sigma_j(\tau_k)\big(\tau_k\tilde{H}_{\sigma_j}\big(\tau_k^{-1}x\big)\big)_t,\quad x\in S_k,\ k\in Ind. \tag{27}$$

Since the values $\tilde{E}_{\sigma_j}(x)$, $\tilde{H}_{\sigma_j}(x)$, and $\sigma_j(\tau_k)$ are matrixes of order $d(\sigma_j)$, relationship (26) can be considered a relation between the columns of the matrix $\tilde{E}_{\sigma_j}(x)$, whereas relationship (27) represents a relation between columns of the matrix $\tilde{H}_{\sigma_j}(x)$. Thus, fields placed in one column of the matrixes $\tilde{E}_{\sigma_j}(x)$, $\tilde{H}_{\sigma_j}(x)$ are related through boundary conditions (26) and (27). Consequently, only $d(\sigma_j)$ independent boundary value problems correspond to each representation σ_j. The dimension of each of them is increased by $d(\sigma_j)$ times. All problems corresponding to the same representation have the same operator. Consequently, we have reduced the initial problem in the whole domain to $s=\sum_{j=1}^{p}d(\sigma_j)$, $s\le N$, independent problems for Maxwell's equations in a fundamental domain Ω_1, which is only $1/N$ part of the whole domain Ω.

Note that the operator of the system of Maxwell's equations does not change because of its invariance with respect to the group G.

As an example we will consider the domain Ω shown in Fig. 1, which has the symmetry group of the square $C_{4\nu}$. This group is a noncommutative one of order 8 and has five irreducible representations: σ_j, $j=1, 2, 3, 4$, are one-dimensional representations and σ_5 is two-dimensional. Hence, a boundary value problem for Maxwell's equations in the domain Ω can be reduced to six problems in the domain Ω_1, which is only $1/8$ of the whole domain Ω.

The problems for the fields $\tilde{E}_{\sigma_j}$, $\tilde{H}_{\sigma_j}$, $j=1,2,3,4$, have an ordinary statement. The problem for fields $\tilde{E}_{\sigma_5}$, $\tilde{H}_{\sigma_5}$ is divided into two independent problems with the same operator: one problem for the fields $(\tilde{E}_{\sigma_j})_{11}$, $(\tilde{E}_{\sigma_j})_{21}$, $(\tilde{H}_{\sigma_j})_{11}$, $(\tilde{H}_{\sigma_j})_{21}$ and one problem for the fields $(\tilde{E}_{\sigma_j})_{12}$, $(\tilde{E}_{\sigma_j})_{22}$, $(\tilde{H}_{\sigma_j})_{12}$, $(\tilde{H}_{\sigma_j})_{22}$. Consequently, for two-dimensional representation σ_5 we have two problems, and the dimension of each of them is doubled because of the coupling of the fields placed in one column of matrixes $\tilde{E}_{\sigma_5}$, $\tilde{H}_{\sigma_5}$ on the cutting boundary $S=S_2\cup S_8$.

As we see from the above example the case of a noncommutative group is more complicated and requires developing appropriate numerical methods for a solution of problems in a domain with noncommutative symmetry group.

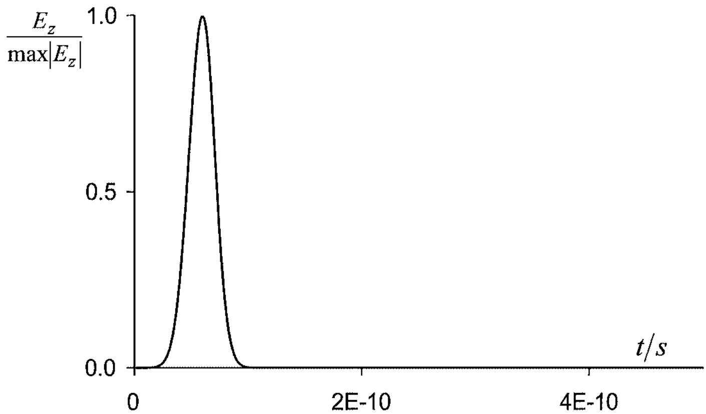

FIG. 2. The form of excitation.

6. A NUMERICAL EXAMPLE OF A SOLUTION OF A BOUNDARY VALUE PROBLEM IN A DOMAIN WITH THE KLEIN SYMMETRY GROUP

To illustrate the method, we consider a scattering problem in two dimensions. We shall assume that the field components do not depend on the z coordinate of a point. We take ε, μ of free space. The only source of our problem is a current source exiting a TM wave, the time dependence of which is shown in Fig. 2. This Gaussian pulse corresponds to a frequency range of 0–30 GHz.

We shall consider the diffraction of a TM wave that exited at point $U = (2\text{ cm},\ 3\text{ cm})$ by a perfectly conducting rectangle, as shown in Fig. 3a. The rectangle has dimensions 2×1 cm. Fields are monitored at four points $p_1 = (1\text{ cm},\ 2.5\text{ cm})$, $p_2 = (1\text{ cm},\ -2.5\text{ cm})$, $p_3 = (-1\text{ cm},\ -2.5\text{ cm})$, and $p_4 = (-1\text{ cm},\ 2.5\text{ cm})$, which are also shown in Fig. 3a.

The rectangle has two planes of symmetry and possesses the Klein symmetry group of order 4. Due to the excitation source the whole problem does not have any symmetry. However, as shown above, it can be reduced to four independent problems in one quarter of the whole domian (see Fig. 3b) with different boundary conditions on $S_2 = \tau_2\Omega_1 \cap \Omega_1$ and $S_4 = \tau_4\Omega_1 \cap \Omega_1$.

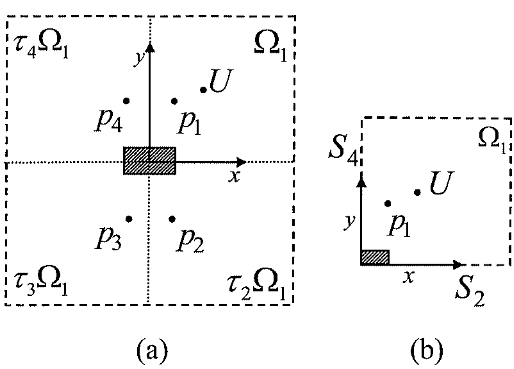

FIG. 3. The geometry for the scattering problem.

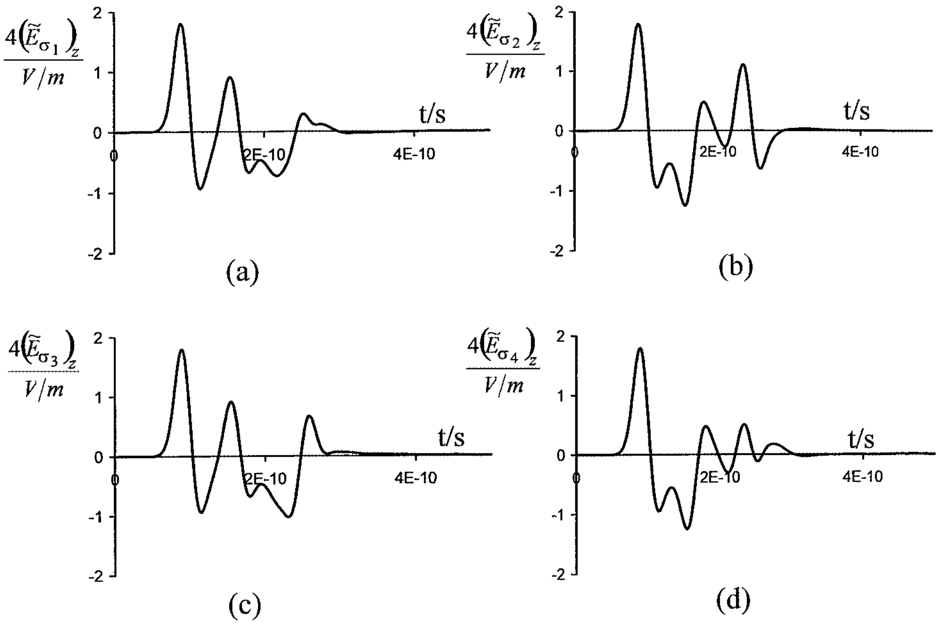

FIG. 4. The solutions in one-quarter of the domain.

The Klein symmetry group is a commutative group and has four unitary nonequivalent one-dimensional representations:

	τ_1	τ_2	τ_3	τ_4
σ_1	1	1	1	1
σ_2	1	1	-1	-1
σ_3	1	-1	-1	1
σ_4	1	-1	1	-1 .

We solve problem (24) for the fields $\tilde{E}_{\sigma_1}$, $\tilde{H}_{\sigma_1}$ in the fundamental domain Ω_1 with boundary condition

$$\left(\tilde{H}_{\sigma_1}(x)\right)_t = 0, \quad x \in S_2 \cup S_4.$$

The component $(\tilde{E}_{\sigma_1})_z$ of this solution at point p_1 is shown in Fig. 4a.

For the fields $\tilde{E}_{\sigma_2}$, $\tilde{H}_{\sigma_2}$ the boundary conditions in the domain Ω_1 have the form

$$\begin{aligned} \left(\tilde{E}_{\sigma_2}(x)\right)_t &= 0, \quad x \in S_2, \\ \left(\tilde{H}_{\sigma_2}(x)\right)_t &= 0, \quad x \in S_4, \end{aligned}$$

The component $(\tilde{E}_{\sigma_2})_z$ at point p_1 is shown in Fig. 4b.

For the fields $\tilde{E}_{\sigma_3}$, $\tilde{H}_{\sigma_3}$ the boundary conditions have the form

$$\begin{aligned} \left(\tilde{H}_{\sigma_3}(x)\right)_t &= 0, \quad x \in S_2, \\ \left(\tilde{E}_{\sigma_3}(x)\right)_t &= 0, \quad x \in S_4, \end{aligned}$$

The component $(\tilde{E}_{\sigma_3})_z$ at point p_1 is shown in Fig. 4c.

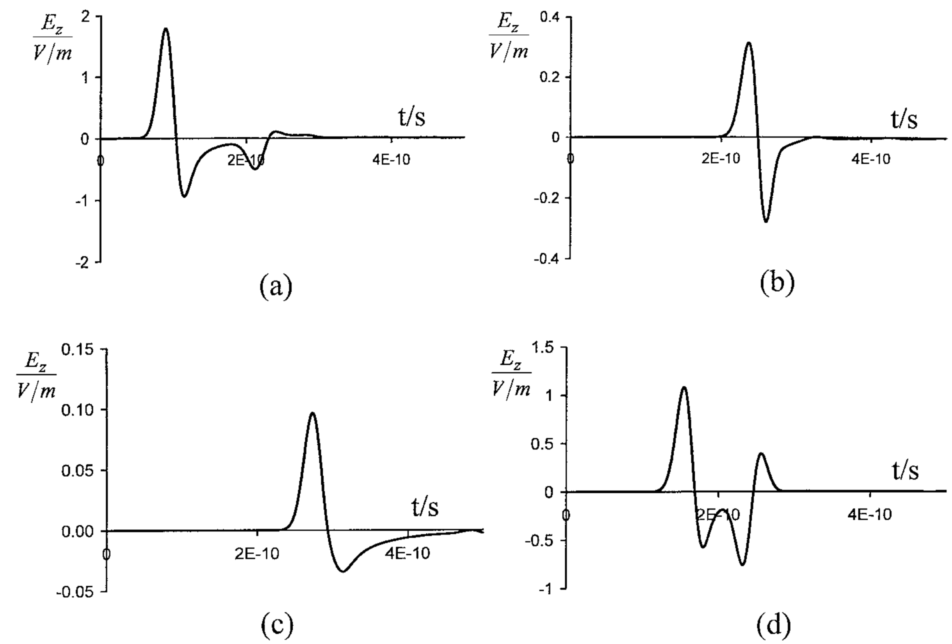

FIG. 5. The full domain solution: (a) at point p_1; (b) at point p_2; (c) at point p_3; (d) at point p_4.

And finally for the fields $\tilde{E}_{\sigma_4}$, $\tilde{H}_{\sigma_4}$ the boundary condition has the form

$$\left(\tilde{E}_{\sigma_4}(x)\right)_t = 0, \quad x \in S_2 \cup S_4.$$

The component $(\tilde{E}_{\sigma_4})_z$ at point p_1 is shown in Fig. 4d.

According to formula (21) we can recover the solution of the whole problem. In Figs. 5a–5d the component E_z at points p_1, p_2, p_3, p_4 is shown correspondingly. It was obtained through the relations

$$\begin{aligned}
E_z &= \left(\tilde{E}_{\sigma_1}\right)_z + \left(\tilde{E}_{\sigma_2}\right)_z + \left(\tilde{E}_{\sigma_3}\right)_z + \left(\tilde{E}_{\sigma_4}\right)_z \quad \text{at } p_1,\\
E_z &= \left(\tilde{E}_{\sigma_1}\right)_z + \left(\tilde{E}_{\sigma_2}\right)_z - \left(\tilde{E}_{\sigma_3}\right)_z - \left(\tilde{E}_{\sigma_4}\right)_z \quad \text{at } p_2,\\
E_z &= \left(\tilde{E}_{\sigma_1}\right)_z - \left(\tilde{E}_{\sigma_2}\right)_z - \left(\tilde{E}_{\sigma_3}\right)_z + \left(\tilde{E}_{\sigma_4}\right)_z \quad \text{at } p_3,\\
E_z &= \left(\tilde{E}_{\sigma_1}\right)_z - \left(\tilde{E}_{\sigma_2}\right)_z + \left(\tilde{E}_{\sigma_3}\right)_z - \left(\tilde{E}_{\sigma_4}\right)_z \quad \text{at } p_4.
\end{aligned}$$

The data shown in Fig. 5 were compared with the solution of the whole problem without taking the symmetry into account. In Fig. 6 the relative deviation $\delta(t) = (E_z(t) - E'_z(t))/\max_t |E'_z(t)|$ is shown, where E'_z is a solution of the whole problem without using of the symmetry and E_z is a solution shown in Fig. 5. Deviation $\delta(t)$ is evaluated at the same points p_1, p_2, p_3, p_4 and Figs. 6a–6d correspond to Figs. 5a–d. The results coincide on the round-off level.

The numerical results presented in this paper were obtained using the program package “CST Microwave Studio 3.0” [8], which uses a finite integration method for solution of Maxwell’s equations in time domains.

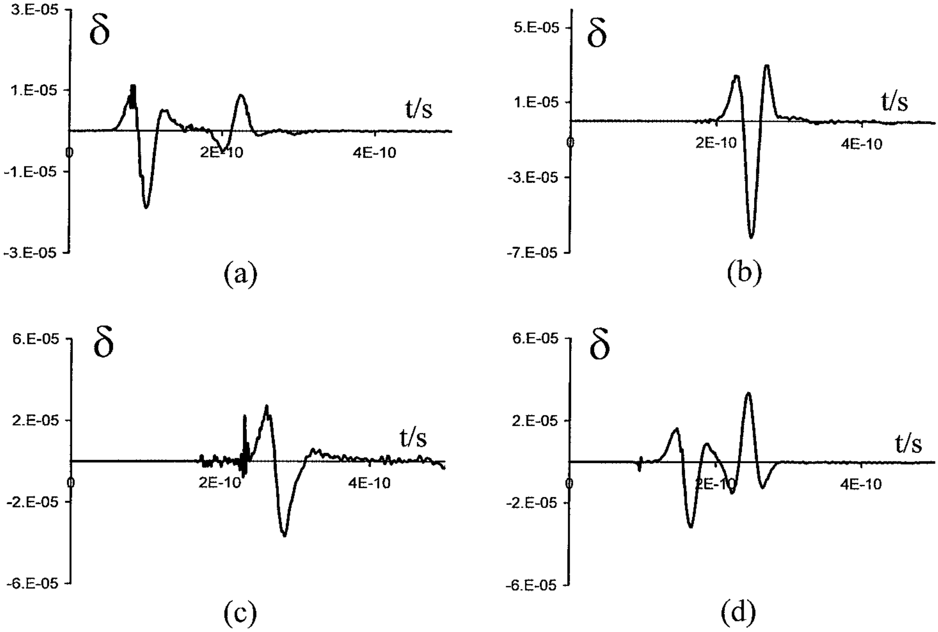

FIG. 6. The comparison of the solutions: (a) at point p_1; (b) at point p_2; (c) at point p_3; (d) at point p_4.

7. CONCLUSIONS

In the paper it is shown that for domains with symmetries, a boundary value problem for Maxwell's equations can be reduced to $\sim N$ independent problems in a fundamental domain, which is only $1/N$ part of the whole domain. In the case of a commutative group the problems retain their dimensions, and so the usual numerical methods of a solution of a boundary value problem for Maxwell's equations can be used. In the case of a noncommutative group, conditions on the cutting boundary are more complicated and the dimension of problems increases. This requires developing special approaches to their numerical solution.

In both cases the dimension of a solution domain is reduced by a factor N. The solution of the reduced problem demands considerably fewer computational resources, and so that approach allows increasing significantly the dimension of the problems which can be treated numerically.

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8. CST Microwave Studio is a trademark of CST GmbH, Buedinger Strasse 2a, 64289 Darmstadt, Germany (www.cst.de).